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The problem of the equilibrium of a helical spring in the non-linear three-dimensional theory of elasticity☆

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Abstract

The problem of the loading of a helical spring by an axial force and a torque is considered using the three-dimensional equations of the non-linear theory of elasticity. The problem is reduced to a two-dimensional boundary-value problem for a plane region in the form of the transverse cross section of the coil of the spring. The solution of the two-dimensional problem obtained enables the equations of equilibrium in the volume of the body and the boundary conditions on the side surface to be satisfied exactly. The boundary conditions at the ends of the spring are satisfied in the integral Saint-Venant sense. The problem of the equivalent prismatic beam in the theory of springs is discussed from the position of the solution of the non-linear Saint-Venant problem obtained. The results can be used for accurate calculations of springs in the non-linear strain region, and also when developing applied non-linear theories of elastic rods with curvature and twisting.

The solution of the Saint-Venant problem for a spring, using the linear theory of elasticity, is described in Refs 1,2.

1. Initial relations

Consider an elastic body which, in the reference configuration, has the form of a helical (spiral) spring with an arbitrary transverse cross section. The body is formed by helical motion along the x_3 axis in the plane of the figure σ, which is situated in the plane passing through the x_3 axis. We will write the equation of the contour $\partial\sigma$, which bounds the region σ, in parametric form: $\rho = \rho(t)$, $\zeta = \zeta(t)$, where ρ is the distance from the x_3 axis and ζ is the distance measured along the x_3 axis. We will call the helical surface formed by the helical motion of the curve $\partial\sigma$ along the x_3 axis the side surface of the spring. When describing the strain of an elastic medium we will use as the Lagrange coordinates non-orthogonal curvilinear coordinates ρ, φ, ζ, connected with the Cartesian coordinates x_1, x_2, x_3 of the unstrained body by the relations

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi, \quad x_3 = \zeta + \mu\varphi \tag{1.1}$$

Here μ is a real number, characterizing the angle of inclination of the coils of the spring to the x_1x_2 plane. When $\mu = 0$ this system of coordinates converts into a system of circular cylindrical coordinates ρ, φ, x_3. The gradient of the

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arbitrary tensor function $\boldsymbol{\Psi}(\rho, \varphi, \zeta)$ in coordinates (1.1) is written as follows:

$$\operatorname{grad}\boldsymbol{\Psi} = \mathbf{g}_1 \otimes \frac{\partial\boldsymbol{\Psi}}{\partial\rho} + \mathbf{g}_2 \otimes \frac{\partial\boldsymbol{\Psi}}{\rho\partial\varphi} + \left(\mathbf{i}_3 - \frac{\mu}{\rho}\mathbf{g}_2\right) \otimes \frac{\partial\boldsymbol{\Psi}}{\partial\zeta}$$
$$\mathbf{g}_1 = \mathbf{i}_1\cos\varphi + \mathbf{i}_2\sin\varphi, \quad \mathbf{g}_2 = -\mathbf{i}_1\sin\varphi + \mathbf{i}_2\cos\varphi \tag{1.2}$$

Here $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are constant unit vectors of the Cartesian coordinates. In the problem considered below, the stress-strain state of the body is the same for all the coils of the spring, and hence we can assume that $0 \le \varphi \le 2\pi$.

Taking the parameters t, φ as Gaussian coordinates, we can write the equation of the side surface of the spring in the form

$$\mathbf{r}(t, \varphi) = \rho(t)\mathbf{g}_1 + \zeta(t)\mathbf{i}_3 + \mu\varphi\mathbf{i}_3 \tag{1.3}$$

where $\mathbf{r} = x_m\mathbf{i}_m$ $(m = 1, 2, 3)$ is the radius vector. Using Eq. (1.3) we can find the unit vector of the normal to the side surface $(\mathbf{g}_3 = \mathbf{i}_3)$

$$\mathbf{n} = \frac{-\rho\zeta'\mathbf{g}_1 - \mu\rho'\mathbf{g}_2 + \rho\rho'\mathbf{i}_3}{\sqrt{(\rho\zeta')^2 + (\mu\rho')^2 + (\rho\rho')^2}} = n_m\mathbf{g}_m \tag{1.4}$$

(the prime denotes a derivative with respect to the variable t). It follows from relation (1.4) that the vector $n_1\mathbf{g}_1 + n_3\mathbf{g}_3$ lies in the plane of the figure σ and is directed along the normal to its boundary $\partial\sigma$. Moreover, the following relation holds

$$n_2 = -\mu\rho^{-1}n_3 \tag{1.5}$$

The system of equations of the statics of a non-linear elastic medium when there are no mass forces has the form[3]

$$\operatorname{div}\mathbf{D} = 0 \tag{1.6}$$

$$\mathbf{D} = dW/d\mathbf{C} = \mathbf{P}\cdot\mathbf{C}; \quad \mathbf{P} = 2dW/d\mathbf{G} \tag{1.7}$$

$$\mathbf{G} = \mathbf{C}\cdot\mathbf{C}^T; \quad \mathbf{C} = \operatorname{grad}\mathbf{R}, \quad \mathbf{R} = X_k\mathbf{i}_k, \quad X_k = x_k + u_k, \quad k = 1, 2, 3 \tag{1.8}$$

and consists of the equilibrium equations for stresses (1.6), constitution relations (1.7) and geometrical relations (1.8). Here div is the divergence operator in Lagrangian coordinates, $\mathbf{C}$ is the strain gradient, X_k are the Cartesian coordinates of the particles of the strained body (Euler coordinates), u_k are the components of the displacement field, $\mathbf{G}$ is the Cauchy measure of strain, $\mathbf{D}$ is the Piola asymmetrical stress tensor, $\mathbf{P}$ is the symmetrical Kirchhoff stress tensor and $W(\mathbf{G})$ is the specific strain potential energy.

We will henceforth assume that the specific energy of the elastic material W, considered as a function of the components $G_{sk} = \mathbf{g}_s \cdot \mathbf{G} \cdot \mathbf{g}_k$ of the Cauchy strain measure in an orthonormalised basis $\mathbf{g}_m$, is explicitly independent of the coordinate φ, but may depend on the coordinates ρ, ζ: $W = W(G_{sk}, \rho, \zeta)$. Such materials will be said to be uniform along the φ coordinate. Such a class of materials includes isotropic elastic media with an arbitrary nonuniformity with respect to the coordinates ρ, ζ, measured in the plane of the azimuthal section σ (i.e. sections of the half-plane $\varphi = \text{const}$) of a coil of the spring, and also certain forms of anisotropic media.

2. Extension – compression and twisting of the spring

We will assume that the side surface of the helical spring is load-free, while a system of forces is applied to each of its ends that is statically equivalent to a longitudinal force F_3, the line of action of which coincides with the spring axis x_3, and a torque with vector $M_3\mathbf{i}_3$. The problem of the equilibrium of an elastic body will be considered as a Saint-Venant problem for a curvilinear rod, in which it is required to construct a solution of Eqs. (1.6)–(1.8), which exactly satisfy the boundary conditions on the side surface of the spring, and approximately, in the Saint-Venant sense, on the ends of the spring. Satisfaction of the boundary conditions on the ends of the spring in the Saint-Venant sense indicates that the stress field acting in one of the outer sections, must have a principal vector equal to $F_3\mathbf{i}_3$, and a principal moment equal to $M_3\mathbf{i}_3$.

To solve this Saint-Venant problem we will consider the following two-parameter family of strains of an elastic body having the geometrical shape described above

$$\begin{aligned} &X_1(\rho, \varphi, \zeta) = \alpha_1 \cos\kappa\varphi - \alpha_2 \sin\kappa\varphi, \quad X_2(\rho, \varphi, \zeta) = \alpha_1 \sin\kappa\varphi + \alpha_2 \cos\kappa\varphi, \\ &X_3(\rho, \varphi, \zeta) = \alpha_3 + \nu\varphi \end{aligned} \tag{2.1}$$

where α_1, α_2, α_3 are functions of the variables ρ, ζ, and $\kappa \neq 0$ and ν are real constants. From the first two relations of (2.1) we obtain the equality

$$X_1^2 + X_2^2 = \alpha_1^2 + \alpha_2^2$$

which shows that the distance of points of the strained side surface from the spring axis x_3 is independent of the φ coordinate. The latter means that for a strain of the form (2.1) the side surface of the body considered remains a helical surface, i.e. the body preserves the shape of a helical spring with the same axis x_3. The diameter and the angle of inclination of the coils change, while the azimuthal section σ of the coils of the spring undergoes a deformation described by the functions α_1, α_2, α_3.

The case $\kappa < 0$ corresponds to the deformation for which the spring is turned inside out, i.e. the circular ring, which is the projection of the body of the spring onto the x_1x_2 plane, is turned inside out. This means that the external and internal circumferences of the ring change roles.

In the special case $\mu = 0$, formulae (2.1) describe the deformation of a sector of a ring, i.e. a curved rod with a circular axis. If, in addition, $\nu = 0$ and $\alpha_2 = 0$, we have pure bending of a curved beam in the x_1x_2 plane. In this case each arc of the circle $\rho = \text{const}$, $\zeta = \text{const}$ converts into an arc of a circle of different radius, which also lies in the horizontal plane, so that the elastic body after deformation keeps the shape of a sector of a solid of revolution.

When $\nu = 0$ relations (2.1) correspond to strains for which the curved rod with axis in the form of a section of a helical line is converted into a circular bar.

By virtue of relations (1.2) and (1.8) the tensor fields of the strain gradient and the Cauchy strain measures, corresponding to the displacement field (2.1), have the form ($k, s = 1, 2, 3$)

$$\begin{aligned} &\mathbf{C}(\rho, \varphi, \zeta) = C_{sk}(\rho, \zeta)\mathbf{g}_s \otimes \mathbf{d}_k, \quad \mathbf{G}(\rho, \varphi, \zeta) = G_{sk}(\rho, \zeta)\mathbf{g}_s \otimes \mathbf{g}_k, \quad G_{sk} = C_{sm}C_{km} \\ &\mathbf{d}_1 = \mathbf{i}_1 \cos\kappa\varphi + \mathbf{i}_2 \sin\kappa\varphi, \quad \mathbf{d}_2 = -\mathbf{i}_1 \sin\kappa\varphi + \mathbf{i}_2 \cos\kappa\varphi, \quad \mathbf{d}_3 = \mathbf{i}_3 \end{aligned} \tag{2.2}$$

$$\begin{aligned} &C_{1k} = \alpha_{k,\rho}, \quad C_{21} = -\frac{(\kappa\alpha_2 + \mu\alpha_{1,\zeta})}{\rho}, \quad C_{22} = \frac{(\kappa\alpha_1 - \mu\alpha_{2,\zeta})}{\rho} \\ &C_{23} = \frac{(\nu - \mu\alpha_{3,\zeta})}{\rho}, \quad C_{3k} = \alpha_{k,\zeta}; \quad \alpha_{k,\rho} \equiv \frac{\partial\alpha_k}{\partial\rho}, \quad \alpha_{k,\zeta} \equiv \frac{\partial\alpha_k}{\partial\zeta} \end{aligned} \tag{2.3}$$

Since, according to Eqs. (2.2) and (2.3), the quantities G_{sk} are independent of the φ coordinate, it follows from Eq. (1.7) that, for a material that is uniform along the φ coordinate, the components of the Kirchhoff stress tensor will be functions solely of the two coordinates ρ and ζ, and hence the Piola stress tensor for a strain of the form (2.1) will have the representation

$$\mathbf{D}(\rho, \varphi, \zeta) = D_{sk}(\rho, \zeta)\mathbf{g}_s \otimes \mathbf{d}_k, \quad k, s = 1, 2, 3 \tag{2.4}$$

Substituting expressions (2.4) into Eq. (1.6), we obtain the scalar form of the equilibrium equations for the Piola stresses

$$\begin{aligned} &(\rho D_{11})_{,\rho} + (\rho D_{31} - \mu D_{21})_{,\zeta} = \kappa D_{22} \\ &(\rho D_{12})_{,\rho} + (\rho D_{32} - \mu D_{22})_{,\zeta} = -\kappa D_{21} \\ &(\rho D_{13})_{,\rho} + (\rho D_{33} - \mu D_{23})_{,\zeta} = 0 \end{aligned} \tag{2.5}$$

Taking into account constitutive relations (1.7) and relations (2.2) and (2.3), we see that Eq. (2.5) represent a system of three scalar equations in three functions of two variables $\alpha_k(\rho, \zeta)$ ($k = 1, 2, 3$). The boundary conditions on the side surface $\mathbf{n}\cdot\mathbf{D}=0$, in accordance with relations (1.4), (1.5) and (2.4), are written in the form

$$n_1 D_{1k} + n_3(D_{3k} - \mu\rho^{-1} D_{2k}) = 0, \quad k = 1, 2, 3 \tag{2.6}$$

Since, according to equality (1.4), the components of the vector of the normal n_1 and n_3 are independent of the φ coordinate, boundary conditions (2.6) do not contain the variable φ and, together with equilibrium Eq. (2.5), form a two-dimensional boundary-value problem for the plane region σ. Hence, assumptions (2.1) regarding the nature of the strain of the elastic medium reduce the initial three-dimensional non-linear problem for a spring to a two-dimensional boundary-value problem for a plane region σ in the form of an azimuthal cross section of a coil of the spring.

Suppose $\alpha_1(\rho, \zeta)$, $\alpha_2(\rho, \zeta)$, $\alpha_3(\rho, \zeta)$ is the solution of boundary-value problem (2.5), (2.6). We will prove that the functions

$$\alpha_1^* = \alpha_1\cos K - \alpha_2 \sin K, \ \alpha_2^* = \alpha_1 \sin K + \alpha_2 \cos K, \ \alpha_3^* = \alpha_3 + L; \ K, L = \text{const} \tag{2.7}$$

also satisfy Eq. (2.5) and boundary conditions (2.6). According to equalities (2.7), replacement (2.7) implies the following replacement of the strain gradient and the Cauchy strain measure ($\mathbf{E}$ is the unit tensor)

$$\begin{gathered}\mathbf{C}_0^* = \mathbf{C}_0\cdot\mathbf{Q}, \quad \mathbf{G}^* = \mathbf{G}, \quad \mathbf{C}_0 \equiv C_{sk}\mathbf{g}_s\otimes\mathbf{g}_k \\ \mathbf{Q} = \mathbf{E}\cos K + (1-\cos K)\mathbf{g}_3\otimes\mathbf{g}_3 - \mathbf{g}_3\times\mathbf{E}\sin K\end{gathered} \tag{2.8}$$

From Eqs. (1.7), (2.4) and (2.8) we obtain the relation

$$\mathbf{D}_0^* = \mathbf{D}_0\cdot\mathbf{Q}, \quad \mathbf{D}_0 \equiv D_{sk}\mathbf{g}_s\otimes\mathbf{g}_k \tag{2.9}$$

It is easily verified that equilibrium Eq. (2.5) and boundary conditions (2.6) are insensitive to the change (2.9). So the functions α_h^* $(k = 1, 2, 3)$ are solutions of the two-dimensional boundary-value problem.

The insensitivity of the boundary-value problem for the region σ to the replacement (2.7) implies that the position of the spring after deformation is defined by this boundary-value problem, apart from rotation around the x_3 axis and a translational displacement along the same axis. One of the methods of eliminating this non-uniqueness of the solution is to make the unknown functions conform to the following additional conditions (everywhere henceforth the integration is carried out over the region σ)

$$\iint[\alpha_3(\rho,\zeta) - \zeta]d\rho d\zeta = 0 \tag{2.10}$$

$$\iint[\cos\Theta(\rho,\zeta) - 1]d\rho d\zeta = 0, \quad \cos\Theta = \frac{\alpha_{1,\zeta} + \alpha_{2,\rho}}{\sqrt{(\alpha_{1,\zeta}+\alpha_{2,\rho})^2 + (\alpha_{2,\zeta} - \alpha_{1,\rho})^2}} \tag{2.11}$$

The geometrical meaning of limitation (2.10) is that the axial displacement of points of the section of a spring coil when $\varphi = 0$ is equal to zero on average over the cross section, while limitation (2.11) denotes that, on average over the cross section $\varphi = 0$, there is no rotation of the material fibres around the x_3 axis.

The solution of the two-dimensional boundary-value problem (2.5), (2.6), (2.10), (2.11) for the plane region σ enables us to satisfy the equilibrium equations in the volume of the body and the boundary conditions on its side surface exactly. To satisfy the boundary conditions on the ends of the spring in the integral Saint-Venant sense we will determine the principal vector $\mathbf{F}$ and the principal moment $\mathbf{M}$ of the forces acting in an arbitrary azimuthal section $\varphi = \text{const}$ of a coil of the spring, undergoing deformation of the form (2.1). Using representation (2.4) we have

$$\begin{gathered}\mathbf{F}(\varphi) = \iint\mathbf{g}_2\cdot\mathbf{D}d\rho d\zeta = F_k\mathbf{d}_k \\ F_k = \iint D_{2k}d\rho d\zeta = \text{const}, \quad k = 1, 2, 3\end{gathered} \tag{2.12}$$

Taking into account the fact that there is no load on the side surface of the body, from the condition for all the forces applied to the part of the coil of the spring between the half-planes $\varphi = \varphi_1$ and $\varphi = \varphi_2$ to balance, we obtain that $\mathbf{F}(\varphi_1) = \mathbf{F}(\varphi_2)$. By virtue of relations (2.2) and (2.12) this leads to the equations

$$\begin{gathered}l_1F_1 - l_2F_2 = 0, \quad l_2F_1 + l_1F_2 = 0 \\ l_1 = \cos\kappa\varphi_1 - \cos\kappa\varphi_2, \quad l_2 = \sin\kappa\varphi_1 - \sin\kappa\varphi_2\end{gathered} \tag{2.13}$$

In view of the fact that the numbers φ_1 and φ_2 are arbitrary, the determinant of the system of Eq. (2.13) for F_1 and F_2 is non-zero. Consequently,

$$F_1 = F_2 = 0, \quad \mathbf{F} = F_3\mathbf{i}_3$$

Since the principal vector $\mathbf{F}$ is parallel to the X_3 axis, the principal moment $\mathbf{M}$ is independent of the choice of the point of application on this axis, which enables us to calculate the moment about the point $X_1 = X_2 = X_3 = 0$. As a result, we obtain

$$\begin{aligned} &\mathbf{M}(\varphi) = -\iint \mathbf{g}_2 \cdot \mathbf{D} \times \mathbf{R} d\rho d\zeta = M_k\mathbf{d}_k, \quad k = 1, 2, 3 \\ &M_1 = \iint (D_{23}\alpha_2 - D_{22}\alpha_3) d\rho d\zeta, \quad M_2 = \iint (D_{21}\alpha_3 - D_{23}\alpha_1) d\rho d\zeta \\ &M_3 = \iint (D_{22}\alpha_1 - D_{21}\alpha_2) d\rho d\zeta \end{aligned} \tag{2.14}$$

Here we have used the representation of the radius vector of a point of the deformed body, which follows from relations (2.1)

$$\mathbf{R} = \alpha_k\mathbf{d}_k + \nu\varphi\mathbf{i}_3$$

According to Eqs. (2.4) and (2.14), the quantities M_k are constant. It follows from the condition for the moments of all the forces applied to the part of the coil of the spring between sections $\varphi = \varphi_1$ and $\varphi = \varphi_2$, to balance, that

$$M_1 = M_2 = 0, \quad \mathbf{M} = M_3\mathbf{i}_3$$

Hence, the realization of deformation (2.1) requires the application to each end of the spring of a system of forces, statically equivalent to a force and a moment acting at a point on the spring axis and directed along this axis. After solving the two-dimensional boundary-value problem (2.5), (2.6), (2.10), (2.11) the value of the force F_3 and the value of the moment M_3 become known functions of the parameters ν and κ.

When $\mu = \nu = \alpha_2 = 0$, i.e. for pure bending of a circular beam, it follows from equalities (2.3) that

$$C_{12} = C_{21} = C_{23} = C_{32} = 0$$

It can then be shown that, for an isotropic material, the following equalities are satisfied

$$D_{12} = D_{21} = D_{23} = D_{32} = 0$$

Hence, the second of the equilibrium Eq. (2.5) and one of the boundary conditions (2.6) are satisfied identically. By virtue of relation (2.12) the resultant $\mathbf{F}$ of the forces acting in any section $\varphi = \text{const}$, is equal to zero. This means that, to maintain a deformation of pure bending of a circular beam, one need only apply bending moments M_3 to its ends.

The potential energy of the deformation of the part of the coil of the spring between the sections $\varphi = \varphi_1$ and $\varphi = \varphi_2$ is given by the formula

$$V = (\varphi_2 - \varphi_1)U, \quad U = \iint W\rho d\rho d\zeta \tag{2.15}$$

Having in mind formulae (2.2) and (2.3), we will consider the functional calculated from the solution of the two-dimensional boundary-value problem (2.5), (2.6), (2.10), (2.11)

$$U(\nu, \kappa) = \iint W[\alpha_k(\rho, \zeta, \nu, \kappa); \nu, \kappa] d\rho d\zeta$$

Here we have taken into account the fact that the solution of the problem on the cross section depends on the parameters ν and κ, while the specific energy W, according to relations (2.2) and (2.3), depends on the parameters ν and κ both explicitly and via the functions α_1, α_2 and α_3.

Theorem. *For an axial force and a torque, acting at the ends of spring, the following energy relations hold*

$$F_3(\nu, \kappa) = \frac{\partial U(\nu, \kappa)}{\partial \nu}, \quad M_3(\nu, \kappa) = \frac{\partial U(\nu, \kappa)}{\partial \kappa} \tag{2.16}$$

Proof. From equalities (1.7), (2.2) and (2.4) we have

$$\begin{aligned}\frac{\partial W}{\partial\kappa} &= \operatorname{tr}\left(\mathbf{D}^T\cdot\frac{\partial\mathbf{C}}{\partial\kappa}\right) = D_{mn}\frac{\partial C_{mn}}{\partial\kappa} + D_{mn}C_{mn}\mathbf{d}_n\cdot\frac{\partial\mathbf{d}_s}{\partial\kappa} = \\ &= D_{mn}\frac{\partial C_{mn}}{\partial\kappa} + (D_{m2}C_{m1} - D_{m1}C_{m2})\varphi, \quad m, n = 1, 2, 3\end{aligned} \tag{2.17}$$

Here we have used the following formulae, which follow from relations (2.2),

$$\frac{\partial\mathbf{d}_1}{\partial\kappa} = \varphi\mathbf{d}_2, \quad \frac{\partial\mathbf{d}_2}{\partial\kappa} = -\varphi\mathbf{d}_1, \quad \frac{\partial\mathbf{d}_3}{\partial\kappa} = 0$$

In view of the symmetry of the Kirchhoff stress tensor **P** in Eq. (1.7), the following equality holds

$$D_{mn}C_{ms} = D_{ms}C_{mn}, \quad m, n, s = 1, 2, 3$$

which makes the second term on the right-hand side of equality (2.17) vanish. From relations (2.3), (2.15) and (2.17) we now obtain ($\alpha_{k,\kappa} = \partial\alpha_k/\partial\kappa$)

$$\begin{aligned}\frac{\partial U}{\partial\kappa} &= \iint\rho D_{mn}\frac{\partial C_{mn}}{\partial\kappa}d\rho d\zeta = \iint(\alpha_1 D_{22} - \alpha_2 D_{21})d\rho d\zeta + \\ &+ \iint\left[\rho D_{1k}\frac{\partial\alpha_{k,\kappa}}{\partial\rho} - \kappa D_{21}\alpha_{2,\kappa} - \mu D_{2k}\frac{\partial\alpha_{k,\kappa}}{\partial\zeta} + \kappa D_{22}\alpha_{1,\kappa} + \rho D_{3k}\frac{\partial\alpha_{k,\kappa}}{\partial\zeta}\right]d\rho d\zeta\end{aligned} \tag{2.18}$$

The second integral on the right-hand side of the second equality of (2.18) vanishes by virtue of equilibrium Eq. (2.5) and boundary conditions (2.6). Referring to formulae (2.14), from (2.18) we obtain the second equality of (2.16). The first equality in the theorem is proved similarly. □

Energy relations, similar to (2.16), arise when there is twisting and tension-compression of a non-linear elastic prismatic rod[4]

$$F(\delta, \psi) = \frac{\partial\Pi(\delta, \psi)}{\partial\delta}, \quad M(\delta, \psi) = \frac{\partial\Pi(\delta, \psi)}{\partial\psi} \tag{2.19}$$

Here δ is the relative longitudinal extension, ψ is the angle of twist, F is the longitudinal force, M is the torque, and Π is the energy per unit length of the rod. Relations (2.19), like formulae (2.16), are exact consequences of the three-dimensional equations of the non-linear theory of elasticity for a prismatic body.

A comparison of relations (2.16) and (2.19) enables us to establish an analogy between a helical spring and a prismatic rod for twisting and tension-compression and to introduce the idea of a rod equivalent to a spring. Suppose l is a certain characteristic dimension of the spring, for example, the length of the middle circle of the ring, which is the projection of a coil of the spring onto the x_1x_2 plane. Then, the potential energy of strain per unit length of the prismatic rod, equivalent to a helical spring in the problem of twisting and tension-compression, is given by the formula

$$\Pi(\delta, \psi) = l^{-1}U(l\delta, l\psi) \tag{2.20}$$

from which we obtain the equalities

$$\frac{\partial\Pi}{\partial\delta} = \frac{\partial U}{\partial\nu}, \quad \frac{\partial\Pi}{\partial\psi} = \frac{\partial U}{\partial\kappa}$$

The problem of the equivalent rod was discussed previously in Ref. 5 within the framework of the applied non-linear theory of springs. In the present paper we have established an analogy between a straight rod and a helical spring on the basis of an exact solution of the non-linear Saint-Venant problem of the twisting and tension-compression of a cylindrical body and a spring.

3. Conversion of the boundary-value problem on a section of a coil of the spring

Since the components of the Piola stress tensor D_{sk} $(s, k=1, 2, 3)$ are expressed non-linearly in terms of the functions α_1, α_2 and α_3 and their derivatives, the two-dimensional boundary-value problem (2.5), (2.6) for the region σ in terms of these functions is a Neumann-type problem with non-linear boundary conditions, which can be converted into a Dirichlet-type problem with linear boundary conditions. To do this we first eliminate the functions α_1, α_2, α_3 from relations (2.3). As a result we obtain the following system of equations

$$\begin{aligned} &\kappa C_{11}-(\rho C_{22})_{,\rho}-\mu C_{32,\rho}=0, \quad \kappa C_{12}+(\rho C_{21})_{,\rho}+\mu C_{31,\rho}=0 \\ &\kappa C_{31}-\rho C_{22,\zeta}-\mu C_{32,\zeta}=0, \quad \kappa C_{32}+\rho C_{21,\zeta}+\mu C_{31,\zeta}=0 \\ &C_{13,\zeta}-C_{33,\rho}=0, \quad \rho C_{23}+\mu C_{33}=\nu \end{aligned} \tag{3.1}$$

Eq. (3.1) will be called the equations of compatibility, since they are the necessary and sufficient conditions for the problem of determining the kinematic variables α_1, α_2 and α_3 to be solvable for unique and differentiable components of the strain gradient C_{ks} (ρ, ζ) $(k, s=1, 2, 3)$ specified in the region σ. It can be shown that, when conditions (3.1) are satisfied, the functions α_1 and α_2 in a simply connected region σ can be found uniquely, while the function α_3 can be found apart from an arbitrary additive constant. If the region σ is multiply connected, the functions α_1 and α_2 are unique, while the function α_3 may be multivalued.

It can be verified by a direct check that the equilibrium equations in Piola stresses (2.5) are satisfied identically by making the substitutions

$$\begin{aligned} &D_{11}=\rho^{-1}\kappa\Phi_{11}, \quad D_{12}=\rho^{-1}\kappa\Phi_{12}, \quad D_{13}=-\rho^{-1}\Phi_{,\zeta} \\ &D_{21}=-\Phi_{12,\rho}-\Phi_{32,\zeta}, \quad D_{22}=\Phi_{11,\rho}+\Phi_{31,\zeta}, \quad D_{23}=\Phi_{23} \\ &D_{31}=\rho^{-1}(\kappa\Phi_{31}-\mu\Phi_{12,\rho}-\mu\Phi_{32,\zeta}), \quad D_{32}=\rho^{-1}(\kappa\Phi_{32}+\mu\Phi_{11,\rho}+\mu\Phi_{31,\zeta}) \\ &D_{33}=\rho^{-1}(\Phi_{,\rho}+\mu\Phi_{23}) \end{aligned} \tag{3.2}$$

The six functions Φ, Φ_{11}, Φ_{12}, Φ_{23}, Φ_{31}, Φ_{32}, in terms of which the general solution of the equilibrium equations is expressed, will be called stress functions. The boundary conditions (2.6) on the boundary $\partial\sigma$ of the azimuthal section of a coil of the spring, can be converted, using (3.2), to the following simple form (s is the length of an arc on the curve $\partial\sigma$)

$$\partial\Phi/\partial s=0, \quad n_1\Phi_{1p}+n_3\Phi_{3p}=0, \quad p=1,2 \tag{3.3}$$

In the case of a simply connected region σ the boundary condition in (3.3) for the function Φ can be replaced, without loss of generality, by the condition $\Phi=0$ on $\partial\sigma$. If the plane region is multiply connected, i.e. contains openings, the function Φ takes constant values on the contours of the openings. These constants differ for different openings and are unknown in advance. Integral relations, similar to relations (2.14) of Ref. 6, serve as additional equations for determining them and express the requirement that the function $\alpha_3(\rho, \zeta)$ should be unique in the multiply connected region.

Using the method described previously,[6,7] we invert the relation $\mathbf{D}(\mathbf{C})$ for the given material, i.e. we express the components of the strain gradient C_{sk} in terms of the components of the Piola stress tensor D_{mn}, and, using relations (3.2), we represent the latter in terms of the stress functions. Then, the six compatibility Eq. (3.1) will be equations in the six stress functions and, together with the linear boundary conditions (3.3), will comprise a Dirichlet-type boundary-value problem.

When formulating the boundary-value problem on a section of a coil of the spring in terms of the stress functions, the integral condition (2.10), which eliminates the possibility of an arbitrary translational displacement of the spring along its axis, obviously becomes unnecessary. In integral condition (2.11) $\cos\Theta$ must be expressed in terms of stress functions. To do this it is sufficient, when taking Eq. (2.3) into account, to replace the derivatives $\alpha_{p,\rho}$ and $\alpha_{p,\zeta}$ $(p=1, 2)$ by the components of the tensor $\mathbf{C}$ and to represent the latter in terms of stress functions using the relation $\mathbf{C}=\mathbf{C}(\mathbf{D})$.

The stress functions enable us to give a variational formulation of the two-dimensional boundary-value problem on the section σ in the form of Castigliano- and Tonti-type variational principles. Using the kinematic variables $\alpha_k(\rho, \zeta)$

($k = 1$, 2, 3) we can formulate Lagrange, Reissner and Hu-Washizu-type variational principles. The formulations and proofs of these variational theorems are similar to those obtained previously[6] in the problem of the equilibrium of a prismatic rod and are not given here.

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